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The Maximum Surplus Before Ruin in the Generalized Erlang(n) Risk Model Perturbed by Diffusion*

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Abstract: The maximum surplus before ruin is an important index of assets in insurance institutions. Considering the important impact of random error factors on the nature of sample paths of the surplus process, which essentially increases difficulties in research, we investigate the distribution of the maximum surplus in generalized Erlang(n) risk model perturbed by diffusion in this paper. We derive a homogeneous integro-differential equation with certain boundary conditions, describing the maximum surplus. Particularly, we can deduce explicit results as long as the individual claim size is rationally distributed. Except for extending a number of results of simple generalized Erlang(n) risk model successfully, our arguments are more practical and the results are more delicate.

Keywords: Sparre Andersen risk process; generalized Erlang(n) claim waiting time; maximum surplus before ruin; diffusion process; integro-differential equation

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1 Introduction

Most literature on ruin theory has concentrated on classical risk models, in which claims occurr as a Poisson process. Andersen[1] assumed the claim process to be occurred as a more general renewal process and derived an integral equation for the corresponding ruin probability. Ever since then, random walks and queuing theory have been used to describe a more general framework of risk models, under which we can deduce explicit results for waiting times or claim severities with the Erlang and phase-type distribution. As the Sparre Anderson risk process can reflect the claim occurrences more properly than classical risk models, it is of great practical value and considerable importance to study and extend this model.

A lot of references (see [2-7] and the references therein) made a thorough study of the Erlang risk model. Further extension to results concerning generalized Erlang(n) claim waiting times have been done by [8-11]. A number of valuable results for survival probability, the surplus immediately prior to ruin, the deficit at ruin and the expected discounted penalty function have been obtained in above works.

To make the model more practical, we introduce uncertain factors into risk models. While the uncertain factor changed the nature of sample paths of the surplus process, which caused

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technique difficulties. Hence, the research of risk models perturbed by diffusion are specially paid attention in risk theory. Gerber first introduced a diffusion process to express additional uncertainty of premium income or aggregate claims. The classical risk model perturbed by diffusion has been studied by many scholars, such as [12-16] and so on.

Li and Garrido[17] considered the expected discounted penalty function for the generalized Erlang(n) risk model perturbed by diffusion. In this paper, with the same model, we mainly study the distribution of the maximum surplus before ruin, which is an important indicator of the assets in insurance institutions. For an insurance company, investigating this maximum can not only grasp its ability of withstanding bankruptcy, but also provide an important basis for carrying out other businesses, such as the dividend and the investment.

Consider a time-continuous Sparre Andersen surplus process perturbed by diffusion

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma W(t), \quad t \geq 0, \tag{1}$$

where $u \geq 0$ is the initial reserve and $c > 0$ is the rate of premium income; $X_i^{'}s$ represent the claims; the counting process $\{N(t) : t \geq 0\}$ denotes the number of claims up to time t; $\{W(t) : t \geq 0\}$ is a standard Brownian motion that is independent of the aggregate claim process

$$S(t) := \sum_{i=1}^{N(t)} X_i$$

and $\sigma > 0$ is the dispersion parameter.

$X_i^{'}s$ are independent and identically distributed (i.i.d.) nonnegative random variables with common distribution P and the density function p. Let $\mu_k = E(X_1^k)$ be the k-th moment of X_1 and

$$\hat{p}(s) = \int_0^\infty \mathrm{e}^{-sx} p(x)\mathrm{d}x$$

the Laplace transform of the density function p. The counting process $\{N(t) : t \geq 0\}$ is defined as $N(t) = \max\{k : T_1 + T_2 + \cdots + T_k \leq t\}$, where the claim waiting times $\{T_i\}_{i=1}^\infty$ are i.i.d. random variables with a common generalized Erlang(n) distribution, i.e. T_i is distributed as the sum of n independent and exponentially distributed r.v.$^{'}s$

$$S_n = V_1 + V_2 + \cdots + V_n, \tag{2}$$

where the V_i may have different exponential parameters $\lambda_i > 0$. In the above model, it is assumed that $\{T_i\}_{i=1}^\infty$ and $\{X_i\}_{i=1}^\infty$ are independent, and $cE(T_i) > E(X_i)$, providing a positive safety loading factor.

For $u \geq 0$, define $T = \inf\{t \geq 0 : U(t) \leq 0\}, (\infty,$ otherwise) to be the ruin time and $\psi(u) = \mathrm{P}(T < \infty \mid U(0) = u)$ to be the ultimate ruin probability.

2 The maximum surplus before ruin

In this section, we consider the distribution of the maximum surplus before ruin. For $0 \leq u < b$, let $\xi(u,b) = \mathrm{P}(\sup_{0\leq t\leq T} U(t) < b,\, T < \infty \mid U(0) = u)$ denote the probability that

ruin occurs from the initial reserve u without the surplus process reaching level b before ruin. Obviously, $\xi(u,b)=0$ for $b\le u$.

In the following, we can derive that $\xi(u,b)$ satisfies an integro-differential equation. Let $\frac{\partial}{\partial u}$ denote the differentiation operator with respect to u and $\mathcal{I}$ the identity operator.

Theorem 2.1 For $0\le u<b$, the probability $\xi(u,b)$ satisfies the following integro-differential equation

$$\Big[\prod_{j=1}^{n}\Big(\mathcal{I}-\frac{c}{\lambda_j}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_j}\frac{\partial^2}{\partial u^2}\Big)\Big]\xi(u,b)=\int_0^u \xi(u-x,b)p(x)\mathrm{d}x+\overline{P}(u) \tag{3}$$

with boundary conditions $\xi(0,b)=1$ and $\xi(b,b)=0$, where $\overline{P}(u)=1-P(u)$.

Proof Fix the number $j=0,1,\cdots,n-1$ in the sum $S_j=V_1+V_2+\cdots+V_j$ in (2) with $S_0=0$, and define, for $0\le u<b$,

$$\begin{aligned}
\xi_{s,j}(u,b) &= P\Big(\sup_{t\le s\le T}U(s)<b,\,T<\infty,\,U(T)<0\,\big|\,S_j=t,\,U(t)=u\Big),\\
\xi_{d,j}(u,b) &= P\Big(\sup_{t\le s\le T}U(s)<b,\,T<\infty,\,U(T)=0\,\big|\,S_j=t,\,U(t)=u\Big),\\
\xi_{j}(u,b) &= P\Big(\sup_{t\le s\le T}U(s)<b,\,T<\infty\,\big|\,S_j=t,\,U(t)=u\Big)\\
&= \xi_{s,j}(u,b)+\xi_{d,j}(u,b).
\end{aligned}$$

Similarly to the proof of Theorem 1 in [17], we have that

$$\Big[\prod_{j=1}^{n}\Big(\mathcal{I}-\frac{c}{\lambda_j}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_j}\frac{\partial^2}{\partial u^2}\Big)\Big]\xi_s(u,b)=\int_0^u \xi_s(u-x,b)p(x)\mathrm{d}x+\overline{P}(u), \tag{4}$$

$$\Big[\prod_{j=1}^{n}\Big(\mathcal{I}-\frac{c}{\lambda_j}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_j}\frac{\partial^2}{\partial u^2}\Big)\Big]\xi_d(u,b)=\int_0^u \xi_d(u-x,b)p(x)\mathrm{d}x. \tag{5}$$

Adding (4) to (5), we obtain the desired integro-differential equation (3). The boundary conditions are obvious: since if $U(0)=0$, ruin occurs immediately, and then $\xi(0,b)=1$; similarly $\xi_j(0,b)=1$, $j=1,2,\cdots,n-1$; moreover $\xi_j(b,b)=0$, $j=0,1,\cdots,n-1$. Then we complete the proof.

Next, for $0\le u\le b$, define $\tau^b=\inf\{t\ge 0: U(t)\ge b\mid U(0)=u\}$ to be the hitting time that the surplus process upcrosses the level b, and $\chi(u,b)=\mathrm{P}(T>\tau^b\mid U(0)=u)$ the probability of the surplus process attaining a given level b without ruin occurring. Noting that $\chi(u,b)=1-\xi(u,b)$.

Corollary 2.2 For $0\le u<b$, the probability $\chi(u,b)$ satisfies the following homogeneous integro-differential equation

$$\Big[\prod_{j=1}^{n}\Big(\mathcal{I}-\frac{c}{\lambda_j}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_j}\frac{\partial^2}{\partial u^2}\Big)\Big]\chi(u,b)=\int_0^u \chi(u-x,b)p(x)\mathrm{d}x \tag{6}$$

with $\chi(0,b)=0$ and $\chi(b,b)=1$.

Proof It is easy to obtain the equation (6) by Theorem 2.1 and $\chi(u,b) = 1 - \xi(u,b)$.

Remark 2.3 Similarly, we define $\chi_j(u,b) = \mathrm{P}(T > \tau^b \mid S_j = t, U(t) = u)$, $j = 0, 1, \cdots, n-1$, and then have, for $m = 1, 2, \cdots, n-1$,

$$\chi_m(u,b) = \Big[\prod_{j=1}^{m}\Big(\mathcal{I} - \frac{c}{\lambda_j}\frac{\partial}{\partial u} - \frac{\sigma^2}{2\lambda_j}\frac{\partial^2}{\partial u^2}\Big)\Big]\chi(u,b)$$

with boundary conditions

$$\chi_m(0,b) = 0, \quad \chi_m(b,b) = 1, \quad m = 0, 1, \cdots, n-1. \tag{7}$$

3 Main results

To solve the integro-differential equation (6) with boundary conditions (7), we consider the solution of the following homogeneous integro-differential equation

$$B(\mathcal{D})v(u) = \int_0^u v(u-x)p(x)\mathrm{d}x, \quad u \geq 0, \tag{8}$$

where $\mathcal{D}$ denotes the differentiation operator and

$$B(\mathcal{D}) = \prod_{j=1}^{n}\Big(\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D} - \frac{\sigma^2}{2\lambda_j}\mathcal{D}^2\Big) = \sum_{k=0}^{2n} B_k\mathcal{D}^k$$

is a $2n$-th order linear differentiation operator.

By the theory of integro-differential equations[7], the general solution of the equation (8) is of the form

$$v(u) = \sum_{i=1}^{2n}\eta_i(b)v_i(u), \quad u \geq 0, \tag{9}$$

where for $i = 1, 2, \cdots, 2n$, $v_i(u)$ are $2n$ linearly independent particular solutions of (8) and $\eta_i(b)$ are any real numbers. Therefore the solution of (6) with boundary conditions (7) is

$$\chi(u,b) = \sum_{i=1}^{2n}\eta_i(b)v_i(u), \quad 0 \leq u \leq b, \tag{10}$$

where $\eta_1(b), \eta_2(b), \cdots, \eta_{2n}(b)$ are determined by the following system of linear equations, for $m = 0, 1, \cdots, n-1$, (by convention $\prod_{m=1}^{0} \cdot = 1$)

$$\Big[\prod_{j=1}^{m}\Big(\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D} - \frac{\sigma^2}{2\lambda_j}\mathcal{D}^2\Big)\Big]\Big[\sum_{i=1}^{2n}\eta_i(b)v_i(u)\Big]\Big|_{u=0} = 0, \tag{11}$$

$$\Big[\prod_{j=1}^{m}\Big(\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D} - \frac{\sigma^2}{2\lambda_j}\mathcal{D}^2\Big)\Big]\Big[\sum_{i=1}^{2n}\eta_i(b)v_i(u)\Big]\Big|_{u=b} = 1. \tag{12}$$

According to above analysis, we know that the solution of (6) heavily depends on the particular solution $v_i(u)$, $i = 1, 2, \cdots, 2n$ of (8). For simplicity, let $w(u)$ denote a particular solution of (8). In the following, we make a detailed discussion about $w(u)$.

3.1 Analysis of the function $w(u)$

At first, we do necessary preparation work. By the equation (12) of [17], we know that

$$B(s) = \hat{p}(s), \quad s \in \mathcal{C}, \tag{13}$$

which is the generalized Lundberg equation. Let $\rho_1, \rho_2, \cdots, \rho_{n-1}$, with $\Re(\rho_i) > 0$ and $\rho_n = 0$ be the n roots of (13). Define the operator T_r of an integrable real function f as

$$T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y) \mathrm{d}y, \quad r \in \mathcal{C}, \quad \Re(r) \geq 0, \quad x \geq 0.$$

This operator has many nice properties, which can be found in [6].

Next, replacing $v(u)$ with $w(u)$ in the equation (8) and then taking Laplace transform on both sides of (8), we have

$$\hat{w}(s) = \frac{d(s)}{B(s) - \hat{p}(s)}, \quad s \in \mathcal{C}, \tag{14}$$

where

$$d(s) = \sum_{j=0}^{2n-1} s^j \sum_{k=j+1}^{2n} B_k w^{(k-1-j)}(0) = \sum_{m=0}^{2n-1} w^{(m)}(0) \sum_{k=m+1}^{2n} B_k s^{k-m-1}$$

is a polynomial of degree $2n-1$. If the $\rho_1, \rho_2, \cdots, \rho_n$ are distinct, by the Lagrange interpolation

$$d(s) = \sum_{j=1}^{n} d(\rho_j) \Big[\prod_{k=1,\ k\neq j}^{n} \frac{s - \rho_k}{\rho_j - \rho_k} \Big],$$

and by the theory of divided differences

$$\begin{aligned} B(s) - \hat{p}(s) &= \Big[\prod_{j=1}^{n} (s - \rho_j) \Big] \big\{ B[\rho_1, \rho_2, \cdots, \rho_n, s] - \hat{p}[\rho_1, \rho_2, \cdots, \rho_n, s] \big\} \\ &= (-1)^n \Big[\prod_{j=1}^{n} (s - \rho_j) \Big] \Big[\frac{\sigma^{2n} \prod_{i=1}^{n} (s + a_i)}{2^n (\prod_{i=1}^{n} \lambda_i)} - T_s T_{\rho_n} T_{\rho_{n-1}} \cdots T_{\rho_1} p(0) \Big], \end{aligned}$$

where $B[\rho_1, \rho_2, \cdots, \rho_n, s]$ and $\hat{p}[\rho_1, \rho_2, \cdots, \rho_n, s]$ are nth divided differences of B and $\hat{p}$ with respect to $\rho_1, \rho_2, \cdots, \rho_n, s$, respectively. The last step holds due to the property of divided difference[9]: since $B[\rho_1, \rho_2, \cdots, \rho_n, s]$ is a polynomial of degree n and the coefficient of s^n is equal to that of s^{2n} in $B(s)$, which is

$$(-1)^n \frac{\sigma^{2n}}{2^n (\prod_{i=1}^{n} \lambda_i)},$$

$B[\rho_1, \rho_2, \cdots, \rho_n, s]$ can be factored as

$$B[\rho_1, \rho_2, \cdots, \rho_n, s] = (-1)^n \frac{\sigma^{2n} \prod_{i=1}^{n} (s + a_i)}{2^n (\prod_{i=1}^{n} \lambda_i)}, \quad s \in \mathcal{C}, \tag{15}$$

where $a_1, a_2, \cdots, a_n$ come in pairs of conjugate complex numbers; there is a close connection between the operator T_r and divided differences, i.e.

$$T_s T_{\rho_n} T_{\rho_{n-1}} \cdots T_{\rho_1} p(0) = (-1)^n \hat{p}[\rho_1, \rho_2, \cdots, \rho_n, s].$$

Consequently, (14) can be rewritten as

$$\hat{w}(s)=\frac{-\frac{2^n(\prod_{i=1}^n\lambda_i)}{\sigma^{2n}}\sum_{j=1}^n d(\rho_j)(\prod_{k=1,\ k\neq j}^n\frac{1}{\rho_k-\rho_j})(\prod_{i=1}^n\frac{1}{s+a_i})(\frac{1}{s-\rho_j})}{1-\frac{2^n(\prod_{i=1}^n\lambda_i)}{\sigma^{2n}}(\prod_{i=1}^n\frac{1}{s+a_i})T_sT_{\rho_n}T_{\rho_{n-1}}\cdots T_{\rho_1}p(0)},\tag{16}$$

inverting (16), we have that

$$w(u)=\int_0^u w(u-y)g(y)\mathrm{d}y+\xi_0+\sum_{i=1}^{n-1}\xi_i\,\mathrm{e}^{\rho_i u}+\sum_{j=1}^n\eta_j\,\mathrm{e}^{-a_j u},\tag{17}$$

where $g(y)=h*l(y)=h_1*\cdots*h_n*l(y)$ with $h_i(y)=\mathrm{e}^{-a_i y}$ for $i=1,2,\cdots,n$, $*$ denoting the convolution and

$$l(y)=\frac{2^n(\prod_{i=1}^n\lambda_i)}{\sigma^{2n}}T_{\rho_n}T_{\rho_{n-1}}\cdots T_{\rho_1}p(y).$$

Furthermore, the coefficients are as follows

$$\begin{aligned}
\xi_0 &= -\frac{2^n(\prod_{i=1}^n\lambda_i)d(0)}{\sigma^{2n}(\prod_{k=1}^{n-1}\rho_k)(\prod_{i=1}^n a_i)},\\
\xi_i &= -\frac{2^n(\prod_{i=1}^n\lambda_i)d(\rho_i)}{\sigma^{2n}[\prod_{k=1,\ k\neq i}^n(\rho_k-\rho_i)][\prod_{l=1}^n(a_l+\rho_i)]},\quad i=1,2,\cdots,n-1,\\
\eta_j &= \frac{2^n(\prod_{i=1}^n\lambda_i)}{\sigma^{2n}}\Big[\prod_{l=1,\ l\neq j}^n\frac{1}{a_l-a_j}\Big]\sum_{r=1}^n d(\rho_r)\Big[\prod_{k=1,\ k\neq r}^n\frac{1}{\rho_k-\rho_r}\Big]\frac{1}{\rho_r+a_j},\quad j=1,2,\cdots,n.
\end{aligned}$$

The equation (17) is a defective renewal equation (see Remark 3.1 below), and the function $w(u)$ determined by (17) is a solution of (8).

Finally, we find $2n$ linearly independent solutions $v_i(u)$, $i=1,2,\cdots,2n$ of (8). Since the above-mentioned function $w(u)$ is uniquely determined by the derivatives of $w(u)$ at 0, i.e. $w^{(k)}(0)$, for $k=0,1,\cdots,2n-1$, we can search for particular solutions $v_i(u)$, $i=1,2,\cdots,2n$ of (8) by determination of the initial conditions.

From [17], we derive that the survival probability $\phi(u)=1-\psi(u)$ satisfies the equation (8) and $\phi'(0)\neq 0$ (see Remark 3.2 below). Since $\phi(0)=0$, we take $v_2(u)=\phi(u)$ as a particular solution of (8). The other particular solutions can be found by specifying the initial conditions as

$$v_i^{(k)}(0)=I\{k=i-1\},\quad k=0,1,\cdots,2n-1,\quad i=1,3,\cdots,2n,$$

where $I\{\cdot\}$ denotes the indicator function. It is easy to prove that the functions $v_i(u)$, $i=1,2,\cdots,2n$ defined as above are linearly independent.

By (17), we show that $v_i(u)$, $i=1,3,\cdots,2n$ all satisfy the defective renewal equation

$$v_i(u)=\int_0^u v_i(u-y)g(y)\mathrm{d}y+\xi_{i,0}+\sum_{k=1}^{n-1}\xi_{i,k}\,\mathrm{e}^{\rho_k u}+\sum_{j=1}^n\eta_{i,j}\,\mathrm{e}^{-a_j u},$$

where $g(y)$ is given as above and the coefficients are as follows

$$\xi_{i,0} = -\frac{2^n(\prod_{i=1}^n \lambda_i)B_i}{\sigma^{2n}(\prod_{k=1}^{n-1}\rho_k)(\prod_{i=1}^n a_i)},$$

$$\xi_{i,k} = \frac{2^n(\prod_{i=1}^n \lambda_i)\sum_{m=0}^{2n-i} B_{m+i}\rho_k^m}{\sigma^{2n}\rho_k[\prod_{j=1,\ j\neq k}^{n-1}(\rho_j-\rho_k)][\prod_{l=1}^n(a_l+\rho_k)]}, \quad k=1,2,\cdots,n-1,$$

$$\eta_{i,j} = \frac{2^n(\prod_{i=1}^n \lambda_i)}{\sigma^{2n}}\Big[\prod_{l=1,\ l\neq j}^{n}\frac{1}{a_l-a_j}\Big]\sum_{r=1}^{n}\sum_{m=0}^{2n-i} B_{m+i}\rho_r^m\Big[\prod_{k=1,\ k\neq r}^{n}\frac{1}{\rho_k-\rho_r}\Big]\frac{1}{\rho_r+a_j}, \quad j=1,2,\cdots,n.$$

3.2 The special case of $w(u)$ for rational claim size

In this section, we consider a special case of the function $w(u)$ when the claim size is rationally distributed. If the density function p is rationally distributed, then w has a rational Laplace transform that can be inverted explicitly by partial fractions as follows. We assume that claim size X_i is rationally distributed, i.e.

$$\hat{p}(s) = \frac{Q_{m-1}(s)}{Q_m(s)}, \quad \Re(s)\in(h_X,\infty), \tag{18}$$

where $m\in\mathcal{N}^+$, $h_X := \inf\{s\in\mathbf{R} : E[e^{-sX_i}]<\infty\}$, Q_m is a polynomial of degree m with leading coefficient 1, Q_{m-1} is a polynomial of degree $m-1$ or less, and Q_m and Q_{m-1} do not have any common zeros. Moreover, since $\hat{p}(s)$ is finite for all s, with $\Re(s)>0$, equation $Q_m(s)=0$ has no roots with positive real parts.

Substituting (18) into (14) yields

$$\hat{w}(s) = \frac{d(s)Q_m(s)}{B(s)Q_m(s)-Q_{m-1}(s)}, \quad s\in\mathcal{C}, \tag{19}$$

where $B(s)Q_m(s)-Q_{m-1}(s)$ is a polynomial of degree $2n+m$ with the leading coefficient

$$(-1)^n\frac{\sigma^{2n}}{2^n(\prod_{i=1}^n\lambda_i)}.$$

Then it can be factored as

$$B(s)Q_m(s)-Q_{m-1}(s) = (-1)^n\frac{\sigma^{2n}}{2^n(\prod_{i=1}^n\lambda_i)}\Big[\prod_{i=1}^n(s-\rho_i)\Big]\Big[\prod_{i=1}^{n+m}(s+R_i)\Big],$$

where $\rho_1,\rho_2,\cdots,\rho_{n-1}$, with $\Re(\rho_i)>0$ and $\rho_n=0$ and $-R_1, -R_2,\cdots,-R_{n+m}$, with $\Re(R_i)>0$, are all the roots of the equation $B(s)Q_m(s)-Q_{m-1}(s)=0$. If $\rho_1,\rho_2,\cdots,\rho_{n-1}$ and $R_1, R_2,\cdots,R_{n+m}$ are all distinct, by partial fractions, the equation (19) can be rewritten as

$$\hat{w}(s) = \frac{\alpha_0}{s}+\sum_{i=1}^{n-1}\frac{\alpha_i}{s-\rho_i}+\sum_{j=1}^{n+m}\frac{\beta_j}{s+R_j}, \tag{20}$$

where

$$\alpha_0 = -\frac{2^n(\prod_{i=1}^n \lambda_i)d(0)Q_m(0)}{\sigma^{2n}(\prod_{i=1}^{n-1}\rho_i)(\prod_{j=1}^{n+m}R_j)},$$

$$\alpha_i = -\frac{2^n(\prod_{i=1}^n \lambda_i)d(\rho_i)Q_m(\rho_i)}{\sigma^{2n}[\prod_{j=1}^{n+m}(R_j+\rho_i)][\prod_{k=1,\ k\neq i}^{n}(\rho_k-\rho_i)]}, \quad i = 1,2,\cdots,n-1,$$

$$\beta_j = \frac{2^n(\prod_{i=1}^n \lambda_i)d(-R_j)Q_m(-R_j)}{\sigma^{2n}[\prod_{i=1}^{n}(R_j+\rho_i)][\prod_{k=1,\ k\neq j}^{n+m}(R_k-R_j)]}, \quad j = 1,2,\cdots,n+m.$$

Taking inverse Laplace transform on the both sides of (20) yields

$$w(u) = \alpha_0 + \sum_{i=1}^{n-1}\alpha_i \mathrm{e}^{\rho_i u} + \sum_{j=1}^{n+m}\beta_j \mathrm{e}^{-R_j u}, \quad u \geq 0. \tag{21}$$

In the following, we find $2n$ linearly independent solutions $v_i(u)$, $i = 1,2,\cdots,2n$ of (8) as given in Section 3.1. By (19), we have that

$$\hat{v}_i(s) = \frac{d_i(s)Q_m(s)}{B(s)Q_m(s) - Q_{m-1}(s)}, \quad s \in \mathcal{C}, \tag{22}$$

where

$$d_i(s) = \sum_{k=i}^{2n} B_k s^{k-i} = \sum_{l=0}^{2n-i} s^l.$$

Making similar discussion as $w(u)$, we obtain

$$\hat{v}_i(s) = \frac{2^n(\prod_{i=1}^n \lambda_i)}{\sigma^{2n}}\frac{d_i(s)Q_m(s)}{[\prod_{j=1}^{n}(\rho_j - s)][\prod_{j=1}^{n+m}(s+R_j)]}. \tag{23}$$

If $\rho_1,\rho_2,\cdots,\rho_{n-1}$ and $R_1,R_2,\cdots,R_{n+m}$ are all distinct, by partial fractions, we obtain

$$\hat{v}_i(s) = \frac{\alpha_{i,0}}{s} + \sum_{k=1}^{n-1}\frac{\alpha_{i,k}}{s-\rho_k} + \sum_{j=1}^{n+m}\frac{\beta_{i,j}}{s+R_j}, \tag{24}$$

where

$$\alpha_{i,0} = -\frac{2^n(\prod_{i=1}^n \lambda_i)B_iQ_m(0)}{\sigma^{2n}(\prod_{i=1}^{n-1}\rho_i)(\prod_{j=1}^{n+m}R_j)},$$

$$\alpha_{i,k} = -\frac{2^n(\prod_{i=1}^n \lambda_i)d_i(\rho_k)Q_m(\rho_k)}{\sigma^{2n}[\prod_{j=1}^{n+m}(R_j+\rho_k)][\prod_{l=1,\ l\neq k}^{n}(\rho_l-\rho_k)]}, \quad k = 1,2,\cdots,n-1,$$

$$\beta_{i,j} = \frac{2^n(\prod_{i=1}^n \lambda_i)d_i(-R_j)Q_m(-R_j)}{\sigma^{2n}[\prod_{l=1}^{n}(R_j+\rho_l)][\prod_{r=1,\ r\neq j}^{n+m}(R_r-R_j)]}, \quad j = 1,2,\cdots,n+m.$$

Inverting (24), we yields that

$$v_i(u) = \alpha_{i,0} + \sum_{k=1}^{n-1}\alpha_{i,k}\mathrm{e}^{\rho_k u} + \sum_{j=1}^{n+m}\beta_{i,j}\mathrm{e}^{-R_j u}, \quad i = 1,3,\cdots,2n, \quad u \geq 0.$$

Remark 3.1 According to the Lemma 1 of [17] and by simple calculations, we obtain that the function $g(y)$ is a defective density function

$$\int_0^\infty g(y)\mathrm{d}y = 1 - \frac{2^n(\prod_{i=1}^n \lambda_i)(c\sum_{i=1}^n \frac{1}{\lambda_i} - \mu_1)}{\sigma^{2n}(\prod_{i=1}^{n-1} \rho_i)(\prod_{i=1}^n a_i)} < 1,$$

which means that (17) is a defective renewal equation.

Remark 3.2 From the conclusion of Theorem 1 in [17], the survival probability $\phi(u)$ satisfies

$$\phi_m(u) = \Big[\prod_{j=1}^m \Big(\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D} - \frac{\sigma^2}{2\lambda_j}\mathcal{D}^2\Big)\Big]\phi(u), \quad m = 1, 2, \cdots, n-1.$$

Let $m = 1$, we have

$$\phi_1(u) = \Big(\mathcal{I} - \frac{c}{\lambda_1}\mathcal{D} - \frac{\sigma^2}{2\lambda_1}\mathcal{D}^2\Big)\phi(u).$$

Taking Laplace transform on both sides of above equation yields

$$\phi'(0) = \frac{2\lambda_1}{\sigma^2}\hat{\phi}_1(s) + \Big(s^2 + \frac{2c}{\sigma^2}s - \frac{2\lambda_1}{\sigma^2}\Big)\hat{\phi}(s).$$

When s is large enough, the right side of above equation is greater than zero, i.e. $\phi'(0) > 0$.

4 The result of $\chi(u, b)$ for $n = 2$ and $\lambda_1 = \lambda_2 = \lambda$

In the following, we consider the Erlang(2) risk model that is perturbed by diffusion, and obtain the expression of $\chi(u, b)$ in terms of the survival probability $\phi(u)$.

Xing and Zhang[18] gave that

$$\hat{\phi}(s) = \frac{\theta\mu_1(s-\rho)(s+\rho+\frac{2c}{\sigma^2})}{\rho(\rho+\frac{2c}{\sigma^2})[B(s)-\hat{p}(s)]}, \tag{25}$$

$$\phi'(0) = \frac{4\lambda^2\theta\mu_1}{\sigma^2\rho(\sigma^2\rho+2c)} > 0, \tag{26}$$

where $\rho > 0$ is the unique positive root of Lundberg equation (13), and

$$\theta = \frac{2c - \lambda\mu_1}{\lambda\mu_1}\lambda\mu_1 > 0$$

is the relative safety loading factor. From Section 2 of [18], it is straightforward to calculate the second derivative at zero, i.e.

$$\phi''(0) = -\frac{8\lambda^2 c\theta\mu_1}{\sigma^4\rho(\sigma^2\rho+2c)} = -\frac{2c}{\sigma^2}\phi'(0).$$

Using the formula (22) and letting

$$\beta := \rho + \frac{2c}{\sigma^2},$$

we have

$$\hat{v}_4(s) = \frac{\sigma^4\rho\beta\hat{\phi}(s)}{4\lambda^2\theta\mu_1(s-\rho)(s+\beta)},$$

which can be inverted as

$$v_4(u) = A\int_0^u \phi(u-y)\mathrm{e}^{\rho y}\mathrm{d}y - A\int_0^u \phi(u-y)\mathrm{e}^{-\beta y}\mathrm{d}y, \tag{27}$$

where

$$A = \frac{\sigma^4\rho\beta}{4\lambda^2\theta\mu_1(\rho+\beta)}.$$

From the discussion in Section 3, one knows that

$$\chi(u,b) = \sum_{i=1}^{4}\eta_i(b)v_i(u),$$

where the coefficients $\eta_1(b)$, $\eta_2(b)$, $\eta_3(b)$, $\eta_4(b)$ are determined by the boundary conditions (11) and (12). We can solve the values of $\eta_i(b)$, for $i = 1,2,3,4$,

$$\eta_1(b) \;=\; \eta_3(b) = 0,$$

$$\eta_2(b) \;=\; \frac{\sigma^2(\rho+\beta)\phi(b) + \sigma^2\rho\beta[\int_0^b \phi(b-y)\mathrm{e}^{\rho y}\mathrm{d}y - \int_0^b \phi(b-y)\mathrm{e}^{-\beta y}\mathrm{d}y]}{\sigma^2(\rho+\beta)\phi^2(b) + [\sigma^2\rho\beta\phi(b) - 2c\phi'(b) - \sigma^2\phi''(b)][\int_0^b \phi(b-y)\mathrm{e}^{\rho y}\mathrm{d}y - \int_0^b \phi(b-y)\mathrm{e}^{-\beta y}\mathrm{d}y]},$$

$$\eta_4(b) \;=\; \frac{-[2c\phi'(b) + \sigma^2\phi''(b)]}{\sigma^2 A(\rho+\beta)\phi^2(b) + A[\sigma^2\rho\beta\phi(b) - 2c\phi'(b) - \sigma^2\phi''(b)][\int_0^b \phi(b-y)\mathrm{e}^{\rho y}\mathrm{d}y - \int_0^b \phi(b-y)\mathrm{e}^{-\beta y}\mathrm{d}y]}.$$

Consequently, $\chi(u,b)$ can be expressed in terms of the survival probability $\phi(u)$. There is the analogous conclusion in Erlang(2) risk model that is given in [7]. As we know, the famous Dickson formula holds in the classical risk model, i.e. $\chi(u,b) = \frac{\phi(u)}{\phi(b)}$. This means that $\chi(u,b)$ is obtained if we have a knowledge of $\phi(u)$.

We remark that when claim sizes are exponentially distributed, there is a closed expression for $\phi(u)$ (see [17]) and then for $\chi(u,b)$.

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带干扰的广义Erlang(n)风险模型破产前资产余额的最大值

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摘 要: 破产前资产余额的最大值是反映保险公司资产实力的重要指标。随机误差因素改变了余额过程的轨道性质，以致于增加了研究上的本质困难。本文研究了带干扰的广义Erlang(n)风险模型破产前资产余额最大值的分布问题。我们推导出破产前资产余额的最大值满足具有一定边界条件的齐次积分微分方程。特别地，当索赔服从有理分布时，我们给出了精确结果。此外，与单纯的广义Erlang(n)风险模型相比较，我们的论证更为复杂结果更为精细，并且推广了那里的结果。

关键词: Sparre Andersen风险模型；广义Erlang(n)索赔间隔；破产前最大余额；扩散过程；积分一微分方程